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ZERO-DIVISOR GRAPH WITH RESPECT TO AN IDEAL

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Let R be a commutative ring with nonzero identity and let I be an ideal of R. The zero-divisor graph of R with respect to I, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set $\{x \in R\backslash I \mid xy \in I$ for some $y \in R\backslash I\}$ with distinct vertices x and y adjacent if and only if $xy \in I$. In the case $I = 0$, $\Gamma_0(R)$, denoted by $\Gamma(R)$, is the zero-divisor graph which has well known results in the literature. In this article we explore the relationship between $\Gamma_I(R) \cong \Gamma_J(S)$ and $\Gamma(R/I) \cong \Gamma(S/J)$. We also discuss when $\Gamma_I(R)$ is bipartite. Finally we give some results on the subgraphs and the parameters of $\Gamma_I(R)$.

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1. INTRODUCTION AND PRELIMINARIES

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero divisors. The *zero-divisor graph*, $\Gamma(R)$, is the graph with vertices $Z(R)^* = Z(R)\backslash\{0\}$, the set of nonzero zero divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. In Beck (1988), the author introduced the concept of a zero-divisor graph of a commutative ring. However, he lets all elements of R be vertices of the graph and his work was mostly concerned with coloring of rings. We adopt the approach used by Anderson and Livingston (1999) and consider only nonzero zero divisors as vertices of the graph. The zero-divisor graph of a commutative ring has been studied extensively by several authors,

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e.g., Beck (1988), Anderson and Livingston (1999), Anderson et al. (2001), Levy and Shapiro (2002), Anderson et al. (2003), Akbari et al. (2003), and Akbari and Mohammadian (2004).

Redmond (2003) introduced the definition of the zero-divisor graph with respect to an ideal. Let I be an ideal of R. The *zero-divisor graph of R with respect to I*, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set $\{x \in R\backslash I \mid xy \in I \text{ for some } y \in R\backslash I\}$ with distinct vertices x and y adjacent if and only if $xy \in I$. Thus if $I = 0$ then $\Gamma_I(R) = \Gamma(R)$, and I is a nonzero prime ideal of R if and only if $\Gamma_I(R) = \emptyset$. Redmond (2003) explored the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$. He gave an example of rings R and S and ideals $I \trianglelefteq R$ and $J \trianglelefteq S$, where $\Gamma(R/I) \cong \Gamma(S/J)$ but $\Gamma_I(R) \ncong \Gamma_J(S)$. Among other things, he showed that for an ideal I of R, $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$. In Section 2, we show that for finite ideals I and J of R and S, respectively, for which $I = \sqrt{I}$ and $J = \sqrt{J}$, if $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$. Also we will show that the converse of this result holds if $|I| = |J|$ (see Theorem 2.2).

For a graph G, the vertices set of G is denoted by $V(G)$. The *degree* of a vertex v in G is the number of edges of G incident with v. We denote by $\delta(G)$ the minimum degree of vertices of G. For any nonnegative integer r, the graph G is called *r-regular* if the degree of each vertex is equal to r. The *girth* of G is the length of a shortest cycle in G and is denoted by $\mathrm{gr}(G)$. If G has no cycles, we define the girth of G to be infinite. An *r-partite* graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite* graph is one in which each vertex is jointed to every vertex that is not in the same subset. The *complete bipartite* (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is jointed by an edge is called a *complete* graph. We use K_n for the complete graph with n vertices. In Section 3, we show that $\Gamma_I(R)$ is a complete bipartite graph provided $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \neq 0$ for prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of R (see Theorem 3.1).

A *clique* of a graph is a maximal complete subgraph and the number of graph vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. In section 4, we show that if I is an ideal of R such that $I = \bigcap_{1\leq i\leq n} \mathfrak{p}_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1\leq i\leq n, i\neq j} \mathfrak{p}_i$ where $\mathfrak{p}_i$'s are prime ideals of R, then $\omega(\Gamma_I(R)) = n$ (see Theorem 4.2).

In this article the notations of graph theory are from Chartrand and Oellermann (1993), and the notations of commutative rings are from Kaplansky (1974).

2. SOME BASIC PROPERTIES OF ZERO-DIVISOR GRAPHS

One of the main questions in the study of zero-divisor graphs is as follows: *Let R and S be two commutative rings. If* $\Gamma(R) \cong \Gamma(S)$*, then do we have* $R \cong S$? Some well-known results on this question are as follows:

(i) If R and S are two finite reduced rings which are not fields, and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see Anderson et al., 2001, Theorem 4.1).
(ii) If R is a finite reduced ring which is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_6$, and S is a ring which is not a local integral domain, and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see Akbari and Mohammadian, 2004, Theorem 5).
(iii) If $R = \prod_{i\in I} F_i$ and $S = \prod_{j\in J} G_j$, where F_i's are finite fields and G_j's are integral domains, and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see Anderson et al., 2003, Theorem 2.1).

Now let I be an ideal of R and J be an ideal of S. It is natural to ask the following question. *If* $\Gamma_I(R) \cong \Gamma_J(S)$, *then do we have* $R/I \cong S/J$? The main purpose of this section is to focus on this question.

A subgraph H of G is called a *spanning* subgraph when $V(G) = V(H)$. A 1-regular spanning subgraph H of G is called a *1-factor* or a *perfect matching* of G. A graph G is *1-factorable* if the edges of G are partitioned into 1-factors of G. Every r-regular bipartite graph is 1-factorable (cf. Chartrand and Oellermann, 1993, p. 192). If the edges of G are partitioned into subgraphs $H_1, \ldots, H_n$, then we write $G \cong H_1 \oplus \cdots \oplus H_n$, and if $H_i \cong H_j$ for all $1 \leq i, j \leq n$, then we write $G \cong nH$, where $H \cong H_i$.

Theorem 2.1. *Let I be a finite ideal of R such that $I = \sqrt{I}$. Then* $\Gamma_I(R) \cong |I|^2\Gamma(R/I)$.

Proof. Let e be the edge of $\Gamma(R/I)$ between the vertices a and b. Since every element of coset $a + I$ is adjacent to every element of coset $b + I$, it is easy to see that there exists a subgraph of $\Gamma_I(R)$, denoted by $H^{(e)}$, which is isomorphic to complete bipartite graph $K_{|I|,|I|}$. On the other hand, by Chartrand and Oellermann (1993, p. 192), we have $K_{|I|,|I|} \cong M_1{}^{(e)} \oplus \cdots \oplus M_{|I|}{}^{(e)}$, where each of $M_i{}^{(e)}$ is a perfect matching of $K_{|I|,|I|}$. Now consider $K_i := \bigoplus_{e\in E(\Gamma(R/I))} M_i{}^{(e)}$ which is a subgraph of $\Gamma_I(R)$. Since $I = \sqrt{I}$, $\Gamma_I(R) \cong K_1 \oplus \cdots \oplus K_{|I|}$. Now the assertion follows from the fact that each K_i is partitioned into $|I|$ edge-disjoint subgraphs, where each of them is isomorphic to $\Gamma(R/I)$. □

Let S be a nonempty set of vertices of a graph G. The *subgraph induced by S* is the maximal subgraph of G with vertex set S, and is denoted by $\langle S\rangle$, that is, $\langle S\rangle$ contains precisely those edges of G joining two vertices in S.

Theorem 2.2. *Let I be a finite ideal of R and let J be a finite ideal of S such that $I = \sqrt{I}$ and $J = \sqrt{J}$. Then the following hold*:

(a) *If* $|I| = |J|$ *and* $\Gamma(R/I) \cong \Gamma(S/J)$, *then* $\Gamma_I(R) \cong \Gamma_J(S)$.
(b) *If* $\Gamma_I(R) \cong \Gamma_J(S)$, *then* $\Gamma(R/I) \cong \Gamma(S/J)$.

Proof. Part (a) is an easy consequence of Theorem 2.1. For proving part (b), let $\varphi: \Gamma_I(R) \longrightarrow \Gamma_J(S)$ be an isomorphism. Now consider $K \subseteq R$ to be a set of distinct representatives of the vertices of $\Gamma(R/I)$. Clearly, the subgraph induced by K is isomorphic to $\Gamma(R/I)$. Now consider the restriction of φ to K. Suppose that $\varphi(K) = K'$ and $\langle K'\rangle = H$. Now, if $a, b \in V(K')$, then $a + J \neq b + J$; otherwise, $a^2 \in J = \sqrt{J}$, and hence $a \in J$, which is a contradiction. Hence, K' is a distinct representation of the vertices of $\Gamma(S/J)$, and hence $\langle K'\rangle = H \cong \Gamma_J(S)$. Therefore, φ induced an isomorphism from $\Gamma(R/I)$ to $\Gamma(S/J)$. □

Note that in Theorem 2.2(a), the condition "$|I| = |J|$" is not superficial, as the following example shows.

Example 2.3. Let $R = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $S = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, and consider $I = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $J = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Hence, $\Gamma(R/I) \cong \Gamma(S/J)$. But by computing the number of edges in each graph we have $\Gamma_I(R) \ncong \Gamma_J(S)$.

The conditions "$I = \sqrt{I}$" and "$J = \sqrt{J}$" on ideals I and J are also necessary in Theorem 2.2 (see Redmond, 2003, Remark 2.3).

Theorem 2.4. *Let I be a nonzero ideal of R and $a \in \Gamma_I(R)$, adjacent to every vertex of $\Gamma_I(R)$. Then $(I : a)$ is a maximal element of the set $\{(I : x) \mid x \in R\}$. Moreover, $(I : a)$ is a prime ideal.*

Proof. Let $V = V(\Gamma_I(R))$. Choose $0 \neq x \in I$. It is easy to see that $a \neq a + x \in \Gamma_I(R)$. Thus $a(a + x) \in I$ and hence $a^2 \in I$. Therefore, $V \cup I = (I : a)$, and so for any $x \in R$, $(I : x)$ is contained in $V \cup I = (I : a)$. Thus the first assertion holds.

Now, we prove that $(I : a)$ is a prime ideal. Let $xy \in (I : a)$ and $x, y \notin (I : a)$. Therefore, $xya \in I$. If $ya \notin I$, then $x \in (I : ya)$. We know that $(I : a) \subseteq (I : ya)$, and therefore, $(I : a) = (I : ya)$. Hence, $x \in (I : a)$, which is a contradiction. □

Theorem 2.5. *Let I be an ideal of R and let S be a clique in $\Gamma_I(R)$ such that $x^2 = 0$ for all $x \in S$. Then $S \cup I$ is an ideal of R.*

Proof. Suppose that $x, y \in S \cup I$. Consider the following three cases.

Case 1. If $x, y \in I$, then $x - y \in S \cup I$.
Case 2. If $x, y \in S$ with $x - y \notin I$, then for all $c \in S$, $c(x - y) \in I$ and hence $S \cup \{x - y\}$ is a clique. Now, since S is a clique, $x - y \in S$.
Case 3. If $x \in I$ and $y \in S$, then $x - y \notin I$, and hence for any $c \in S$, $c(x - y) \in I$. Therefore, $x - y \in S$.

Now, let $x \in S \cup I$ and $r \in R$. Suppose that $r, x \notin I$. If $rx \in I$, then $rx \in S \cup I$. If $rx \notin I$, since for any $c \in S$, $rxc \in I$, we have $rx \in S$. □

Theorem 2.6. *Let I be an ideal of R and consider $S = \sqrt{I} \backslash I$. If S is a nonempty set, then $\langle S \rangle$ is connected.*

Proof. Let x, $y \in S$. If $xy \in I$, then the result is obtained. Suppose that $xy \notin I$, where x^n, $y^m \in I$ and x^{n-1}, $y^{m-1} \notin I$. Hence, the path

$$x — x^{n-1} — xy — y^{m-1} — y$$

is a path of length four from x to y. □

Corollary 2.7. *Suppose either N is the nil radical of R, or is a nilpotent ideal of R. If N is nontrivial, then $\langle N \backslash \{0\} \rangle$ is a connected subgraph of $\Gamma(R)$.*

3. COMPLETE *r*-PARTITE GRAPH

It is easy to see that if I is a prime ideal of R, then we have $\Gamma_I(R) = \emptyset$. In the following, we show that if $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals of R, then $\Gamma_I(R)$ is a complete bipartite graph. In Section 4, we study the girth and the clique number of $\Gamma_I(R)$ for $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$, where $\mathfrak{p}_i$'s are prime ideals of R.

Theorem 3.1. *Let I be a nonzero ideal of R. Then the following hold*:

(a) *If $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals of R and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \neq 0$, then $\Gamma_I(R)$ is a complete bipartite graph.*
(b) *If $I \neq 0$ is an ideal of R for which $I = \sqrt{I}$, then $\Gamma_I(R)$ is a complete bipartite graph if and only if there exist prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of R such that $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$.*

Proof. (a): Let $a, b \in R\backslash I$ with $ab \in I$. Then $ab \in \mathfrak{p}_1$ and $ab \in \mathfrak{p}_2$. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, we have $a \in \mathfrak{p}_1$ or $b \in \mathfrak{p}_1$ and $a \in \mathfrak{p}_2$ or $b \in \mathfrak{p}_2$. Therefore, suppose $a \in \mathfrak{p}_1\backslash\mathfrak{p}_2$ and $b \in \mathfrak{p}_2\backslash\mathfrak{p}_1$. Thus, $\Gamma_I(R)$ is a complete bipartite graph with parts $\mathfrak{p}_1\backslash\mathfrak{p}_2$ and $\mathfrak{p}_2\backslash\mathfrak{p}_1$.

(b): Suppose that the parts of $\Gamma_I(R)$ are V_1 and V_2. Set $\mathfrak{p}_1 = V_1 \cup I$ and $\mathfrak{p}_2 = V_2 \cup I$. It is clear that $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$. We now prove that $\mathfrak{p}_1$ is an ideal of R. To show this let $a, b \in \mathfrak{p}_1$.

Case 1. If $a, b \in I$, then $a - b \in I$ and so $a - b \in \mathfrak{p}_1$.
Case 2. If $a, b \in V_1$, then there is $c \in V_2$ such that $ca \in I$ and $cb \in I$. So, $c(a - b) \in I$. If $a - b \in I$, then $a - b \in \mathfrak{p}_1$. Otherwise, $a - b \in V_1$, which implies $a - b \in \mathfrak{p}_1$.
Case 3. If $a \in V_1$ and $b \in I$, then $a - b \notin I$, so there is $c \in V_2$ such that $c(a - b) \in I$. This implies that $a - b \in V_1$, and so $a - b \in \mathfrak{p}_1$.

Now let $r \in R$ and $a \in \mathfrak{p}_1$.

Case 1. If $a \in I$, then $ra \in I$ and so $ra \in \mathfrak{p}_1$.
Case 2. If $a \in V_1$, then there exists $c \in V_2$ such that $ca \in I$. So, $c(ra) \in I$. If $ra \in I$, then $ra \in \mathfrak{p}_1$ and if $ra \notin I$, then $ra \in V_1$ which implies $ra \in \mathfrak{p}_1$. Therefore, $\mathfrak{p}_1 \trianglelefteq R$.

We now prove $\mathfrak{p}_1$ is prime. For proving this let $ab \in \mathfrak{p}_1$ and $a, b \notin \mathfrak{p}_1$. Since $\mathfrak{p}_1 = V_1 \cup I$, $ab \in I$ or $ab \in V_1$, and so in any case there exists $c \in V_2$ such that $c(ab) \in I$. Thus $a(cb) \in I$. If $cb \in I$, then by the definition of $\Gamma_I(R)$ we have $b \in V_1$, that is a contradiction. Hence, $cb \notin I$ and so $cb \in V_1$. Therefore, $c^2b \in I$. Since $I = \sqrt{I}$, $c^2 \notin I$. Hence, $c^2 \in V_2$. So $b \in V_1$ which is a contradiction. Therefore, $\mathfrak{p}_1$ is a prime ideal of R. □

Note that if we consider $R = \mathbb{Z}_8$ and $I = \langle 4 \rangle$, then it is easy to see that $\Gamma_I(R)$ is bipartite, but I cannot be written as the intersection of two prime ideals. Therefore, the converse of Theorem 3.1(a) is not valid in general. Hence, the condition "$I = \sqrt{I}$" on ideal I is not superficial in Theorem 3.1(b).

Theorem 3.2. *Let I be a nonzero proper ideal of R. If $\Gamma_I(R)$ is a complete r-partite graph, $r \geq 3$, then at most one of the parts has more than one vertex.*

Proof. Assume that $V_1, \ldots, V_r$ are parts of $\Gamma_I(R)$. Let V_t and V_s have more than one element. Choose $x \in V_t$ and $y \in V_s$. Let V_l be a part of $\Gamma_I(R)$ such that $V_l \neq V_t$ and $V_l \neq V_s$. Let $z \in V_l$. Since $\Gamma_I(R)$ is a complete r-partite graph, $(I : x) = (\bigcup_{1\leq i\leq r, i\neq t} V_i) \cup I$, $(I : y) = (\bigcup_{1\leq i\leq r, i\neq s} V_i) \cup I$, and $(I : z) = (\bigcup_{1\leq i\leq r, i\neq l} V_i) \cup I$. Therefore, $(I : z) \subseteq (I : x) \cup (I : y)$, and so we have $(I : z) \subseteq (I : x)$ or $(I : z) \subseteq (I : y)$. Let $(I : z) \subseteq (I : x)$ and choose $x' \in V_t$ such that $x' \neq x$. Then we have $x' \in (I : z)\backslash (I : x)$. This is a contradiction. □

4. GIRTH AND CLIQUE NUMBER

In this section we study the girth and the clique number of $\Gamma_I(R)$, when I is an intersection of prime ideals.

Theorem 4.1. *Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be prime ideals of R and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$. Then either $\mathrm{gr}(\Gamma_I(R)) = 4$ or $R/I \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Proof. If $|\mathfrak{p}_1 \backslash \mathfrak{p}_2| = 1$ and $|\mathfrak{p}_2 \backslash \mathfrak{p}_1| \geq 2$, then $\Gamma_I(R)$ is a star graph and so has a cut point. This is a contradiction by Redmond (2003, Theorem 3.2). Therefore, this case cannot happen. The case $|\mathfrak{p}_2 \backslash \mathfrak{p}_1| = 1$ and $|\mathfrak{p}_1 \backslash \mathfrak{p}_2| \geq 2$ is similar. So there are two other possibilities.

Case 1. $|\mathfrak{p}_i \backslash \mathfrak{p}_j| \geq 2$ for $i \neq j$ and $1 \leq i,\ j \leq 2$. In this case, Theorem 3.1 implies that $\mathrm{gr}(\Gamma_I(R)) = 4$.

Case 2. $|\mathfrak{p}_i \backslash \mathfrak{p}_j| < 2$ for $i \neq j$ and $1 \leq i,\ j \leq 2$. In this case, there is $x \in R$ for which $\mathfrak{p}_1 \backslash \mathfrak{p}_2 = \{x\}$ and so $\mathfrak{p}_1 = \{x\} \cup I$. For any $r \in R \backslash \mathfrak{p}_2$ we have $rx \in \mathfrak{p}_1 \backslash I$ and so $rx = x$. Therefore, $(1-r)x = 0 \in \mathfrak{p}_2$ and hence $(1-r) \in \mathfrak{p}_2$. Thus $|R/\mathfrak{p}_2| = 2$. That implies $\mathfrak{p}_2$ is a maximal ideal of R and $R/\mathfrak{p}_2 \cong \mathbb{Z}_2$. But $\mathfrak{p}_1 + \mathfrak{p}_2 = R$, so that implies $R/I \cong R/\mathfrak{p}_1 \times R/\mathfrak{p}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

Theorem 4.2. *Let I be an ideal of R such that $I = \bigcap_{1 \leq i \leq n} \mathfrak{p}_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1 \leq i \leq n, i \neq j} \mathfrak{p}_i$ where $\mathfrak{p}_i$'s are prime ideals of R. Then $\omega(\Gamma_I(R)) = n$.*

Proof. Consider $x_j \in \bigcap_{1 \leq i \leq n, i \neq j} \mathfrak{p}_i \backslash \mathfrak{p}_j$. It is easy to see that $X = \{x_1, \ldots, x_n\}$ is a clique in $\Gamma_I(R)$. Hence, $\omega(\Gamma_I(R)) \geq n$ and so it is sufficient to show that $\omega(\Gamma_I(R)) \leq n$. In order to do this, we use induction on n. For $n = 2$, by Theorem 3.1, $\Gamma_I(R)$ is a bipartite graph and hence $\omega(\Gamma_I(R)) = 2$. Suppose $n > 2$ and the result is true for any integer less than n. Let $I = \bigcap_{1 \leq i \leq n} \mathfrak{p}_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1 \leq i \leq n, i \neq j} \mathfrak{p}_i$. Let $\{x_1, \ldots, x_m\}$ be a clique in $\Gamma_I(R)$. Hence, $x_1 x_j \in \bigcap_{1 \leq i \leq n} \mathfrak{p}_i$ for any $2 \leq j \leq m$. Without loss of generality, suppose that $x_1 \notin \mathfrak{p}_1$. Therefore, $x_2, \ldots, x_m \in \mathfrak{p}_1$, so $x_2, \ldots, x_m \notin \bigcap_{2 \leq i \leq n} \mathfrak{p}_i$. Let $J = \bigcap_{2 \leq i \leq n} \mathfrak{p}_i$. Hence, $\{x_2 \ldots, x_m\}$ is a clique in $\Gamma_J(R)$. Therefore, $m - 1 \leq n - 1$, and the result is obtained. □

Corollary 4.3. *The following hold*:

(a) *If $I = \bigcap_{1 \leq i \leq n} \mathfrak{p}_i \neq 0$ and $J = \bigcap_{1 \leq j \leq m} \mathfrak{q}_j$ where $\mathfrak{p}_i$'s and $\mathfrak{q}_j$'s are prime ideals such that $\Gamma_I(R) = \Gamma_J(R)$, then $m = n$.*
(b) *If for any $\mathfrak{p} \in \mathrm{Min}(R)$, $\mathfrak{p}$ is a finitely generated ideal, then $\omega(\Gamma_{\mathrm{nil}(R)}(R)) = |\mathrm{Min}(R)|$ (which is finite by the main theorem of Anderson,* 1994).
(c) *If R is a semi-local ring and not local, then $\omega(\Gamma_{J(R)}(R)) = |\mathrm{Max}(R)|$.*
(d) *If n is a square-free integer, then $\omega(\Gamma_{n\mathbb{Z}}(\mathbb{Z})) = k$, where k is the number of primes in the decomposition of n into primes.*

Theorem 4.4. *Let I be an ideal of R. Suppose either I is a primary ideal of R that is not prime and $|I| \geq 3$, or $|\mathrm{Ass}(R/I)| \geq 3$. Then $\mathrm{gr}(\Gamma_I(R)) = 3$.*

Proof. For the first case, let $a, b \in R \backslash I$ such that $ab \in I$. Then there exists $n \in \mathbb{N}$ such that $b^n \in I$, so we can choose $t \in \mathbb{N}$ for which $b^t \in I$ and $b^{t-1} \notin I$. Since a, $b^{t-1} \notin I$, we have the chain

$$a - b - b^{t-1} - a$$

in the graph $\Gamma_I(R)$. Therefore, $\mathrm{gr}(\Gamma_I(R)) = 3$.

For the second case, $|\mathrm{Ass}(R/I)| \geq 3$ implies that $\mathrm{gr}(\Gamma(R/I)) = 3$ (see Akbari et al., 2003, Corollary 2.2), and hence $\mathrm{gr}(\Gamma_I(R)) = 3$. □

In the above theorem, one of the conditions "$|I| \geq 3$" or "$|\mathrm{Ass}(R/I)| \geq 3$" are necessary. To see this, for example let $R = \mathbb{Z}_8$ and consider $I = \langle 4 \rangle$; and note that we have $|\mathrm{Ass}(R/I)| = 1$ and $\mathrm{gr}(\Gamma(R/I)) = \infty$.

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REFERENCES

Akbari, S., Mohammadian, A. (2004). On the zero-divisor graph of a commutative ring. *J. Algebra* 274(2):847–855.

Akbari, S., Maimani, H. R., Yassemi, S. (2003). When a zero-divisor graph is planar or a complete r-partite graph. *J. Algebra* 270(1):169–180.

Anderson, D. D. (1994). A note on minimal prime ideals. *Proc. Amer. Math. Soc.* 122(1):13–14.

Anderson, D. F., Livingston, P. S. (1999). The zero-divisor graph of a commutative ring. *J. Algebra* 217(2):434–447.

Anderson, D. F., Frazier, A., Lauve, A., Livingston, P. S. (2001). *The Zero-Divisor Graph of a Commutative Ring II*. Lecture Notes in Pure and Appl. Math. 220. New York: Dekker.

Anderson, D. F., Levy, R., Shapiro, J. (2003). Zero-divisor graphs, von Neumann regular rings, and boolean algebras. *J. Pure Appl. Algebra* 180(3):221–241.

Beck, I. (1988). Coloring of commutative rings. *J. Algebra* 116(1):208–226.

Chartrand, G., Oellermann, O. R. (1993). *Applied and Algorithmic Graph Theory*. New York: McGraw-Hill, Inc.

Kaplansky, I. (1974). *Commutative Rings*. Chicago, Ill.-London: The University of Chicago Press.

Levy, R., Shapiro, J. (2002). The zero-divisor graph of von Neumann regular rings. *Comm. Algebra* 30(2):745–750.

Redmond, S. P. (2003). An ideal-based zero-divisor graph of a commutative ring. *Comm. Algebra* 31(9):4425–4443.